SPHERICAL COMPLEXES AND RADIAL PROJECTIONS OF POLYTOPES

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ABSTRACT

Using the techniques of Gale diagrams a simple criterion is given for determining when a given spherical complex on $S^{n-1} \subset E^n$ is the radial projection, from the centre of S^{n-1}, of a convex polytope. Previously a criterion was known only for the case $n = 2$.

Let S^{n-1} be the unit $(n-1)$-sphere in E^n centred on the origin o. A strongly convex (closed) spherical polytope on S^{n-1} is defined to be the intersection of a finite set of hemispheres that does not contain any pair of antipodal points. By a *spherical complex* on S^{n-1} we mean a finite collection $\mathscr{C}$ of such polytopes (called *cells*) that satisfies the following three conditions:

i) Every face of a cell of $\mathscr{C}$ is a cell of $\mathscr{C}$.

ii) The intersection of any two cells of $\mathscr{C}$ is a face (proper or improper) of both of them.

iii) The union of all the cells of $\mathscr{C}$ is S^{n-1}.

(Cairns [1], who was interested in the case $n = 3$, called such a complex a *geodesic complex* since its 1-cells are minor arcs of great circles on S^{n-1}.)

A simple way to construct a spherical complex is as follows: Let P be an n-polytope (an n-dimensional convex polytope) in E^n such that $o \in \text{int } P$. Consider the radial projection f: $bd\, P \rightarrow S^{n-1}$ in which, for each $x \in bd\, P, f(x) = r(x) \cap S^{n-1}$, where $r(x)$ is the ray with end-point o containing x. Then the image under f of each proper face of P is a strongly convex spherical polytope, and so the image of the boundary complex of P is a spherical complex $\mathscr{C}(P)$. This fact follows immediately from the elementary properties of convex polytopes given, for example, in [2] and [3], to which the reader is referred for further information on this topic.

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It is evident that there exist spherical complexes other than those of the type $\mathscr{C}(P)$. It is therefore of some interest to solve the following problem.

(1) *Find necessary and sufficient conditions for a given spherical complex $\mathscr{C}$ to be the image of the boundary complex of a convex polytope under radial projection.*

This question was discussed at some length by Supnick [5] who obtained a criterion in terms of the solubility of a system of linear inequalities. Here we shall obtain an alternative solution which resembles Supnick's in that it involves linear inequalities, but differs in that it is, computationally and conceptually, much simpler, and applies to general spherical complexes. (Supnick's solution relates only to simplicial spherical complexes on the 2-sphere.) Our discussion depends on a result recently obtained by the author in connection with an investigation into the properties of positive bases [4], and uses the techniques of representations and diagrams.

Let $\mathscr{C}$ be a spherical complex on S^{n-1}, and let $V = \{v_1, \cdots, v_s\}$ be the set of vertices of $\mathscr{C}$. We construct a linear representation $\bar{V} = \{\bar{v}_1, \cdots, \bar{v}_s\} \subset E^{s-n}$ (see [4, §1]). Since the cells of $\mathscr{C}$ are strongly convex each open hemisphere of S^{n-1} contains at least one point v_i, and so V positively spans E^n. We deduce [4, Corollary to Theorem 1] that $o \notin \mathrm{conv}\ \bar{V}$, so that pos $\bar{V}$ is a proper polyhedral cone with a single vertex o. Let H be any hyperplane not containing o and such that $H \cap \mathrm{pos}\ \bar{V}$ is an $(s-n-1)$-polytope. For each i write $\hat{v}_i = r(\bar{v}_i) \cap H$, and then $\hat{V} = \{\hat{v}_1, \cdots, \hat{v}_s\}$ may be called a *diagram* for the set V. We note that $\hat{V}$ is determined only within a projective transformation permissible for conv $\hat{V}$. The basic result we shall need is a slight modification of Theorem 5 of [4].

(2) *Let P be any polytope whose s vertices lie one on each of the rays $r(v_i)$, $i = 1, \cdots, s$. Then $\hat{V}$ is a Gale diagram of P if we select some suitable point $z \in H$ as origin.*

Conversely, if we select any $z \in H$ in such a way that every half-space bounded by a hyperplane through z contains at least two points of $\hat{V}$, then $\hat{V}$ with z as origin is a Gale diagram of some polytope with s vertices, one of which lies on each of the rays $r(v_i)$.

The proof of this result is exactly the same as that given for Theorem 5 of [4]. We remark that in the case where $\{v_1, \cdots, v_s\}$ is a positive basis then z may be selected as any point of int conv $\hat{V}$ for then the condition that every half-space contains at least two points of $\hat{V}$ is automatically satisfied. It will be seen that (2) provides a simple and straightforward method of determining the combinatorial structure of every n-polytope whose vertices project radially into the vertices of a given spherical complex $\mathscr{C}$, and hence we can obtain an answer to our problem (1).

We proceed as follows. Let $\mathscr{C} \subseteq S^{n-1}$ be a given spherical complex, and suppose, renumbering if necessary, that $\{v_1, \cdots, v_t\}$ is the complete set of vertices of an $(n-1)$-cell C_i of $\mathscr{C}$. Write $Z_i =$ relint conv $\{\hat{v}_{t+1}, \cdots, \hat{v}_s\}$ and let $\mathscr{Z}$ be the family of all such sets Z_i corresponding to the $(n-1)$-cells of $\mathscr{C}$. Then we have the following result.

(3) THEOREM. *A spherical complex $\mathscr{C}$ is a radial projection of the boundary complex of a convex polytope P if and only if*

$$\cap \mathscr{Z} \neq \emptyset.$$

If there exists a point $z \in \cap \mathscr{Z}$ then $\hat{V}$, with z as origin, is a Gale diagram of P.

We illustrate (3) with two very simple examples. Consider the set of six points.

a	$(1,0,1)$	b	$(-\frac{1}{2},\frac{1}{2}\sqrt{3},\ 1)$	c	$(-\frac{1}{2},-\frac{1}{2}\sqrt{3},\ 1)$
d	$(1,0,-1)$	e	$(-\frac{1}{2},\frac{1}{2}\sqrt{3},-1)$	f	$(-\frac{1}{2},-\frac{1}{2}\sqrt{3},-1)$

on the sphere $\sqrt{2}\, S^2$. Then a diagram of this set consists of six points in the plane arranged as the vertices of a regular hexagon (or, more generally, of any hexagon which is a permissible projective image of a regular hexagon).

Let $\mathscr{C}_1$ be a spherical complex with this set of six points as vertices, and five 2-cells with vertices

abc, *def,* *bcef,* *cafd,* *abde,*

see Fig. 1. Then $\mathscr{Z}$ consists of two open triangles (int conv $\{\hat{d},\hat{e},\hat{f}\}$, int conv $\{\hat{a},\hat{b},\hat{c}\}$) and three open line segments (relint conv $\{\hat{a},\hat{d}\}$, relint conv $\{\hat{b},\hat{e}\}$, relint conv $\{\hat{c},\hat{f}\}$), see Fig. 2. The centre of the hexagon belongs to all the sets Z_i so we take this point as z. Statement (3) implies that $\mathscr{C}_1$ is the radial projection of a convex polytope, and the Gale diagram $\hat{V}$, with z as origin, shows that this polytope is combinatorially isomorphic to a triangular prism.

Now let $\mathscr{C}_2$ be the same as $\mathscr{C}_1$ except that there is an additional edge (1-cell) ·oining b to d, see Fig. 3. Then $\mathscr{C}_2$ has six 2-cells, namely those of $\mathscr{C}_1$ except that the cell with vertices *abde* is replaced by two triangular cells with vertices *abd* and *bde*. The open triangle int conv $\{\hat{c}, \hat{e}, \hat{f}\}$ and the two open line segments relint conv $\{\hat{a}, \hat{d}\}$ and relint conv $\{\hat{b}, \hat{e}\}$ are three of the sets of $\mathscr{Z}$. As these three sets have empty intersection (see Fig. 4), therefore $\cap \mathscr{Z} = \emptyset$, and we deduce from (3) that $\mathscr{C}_2$ is not the radial projection of any convex polytope.

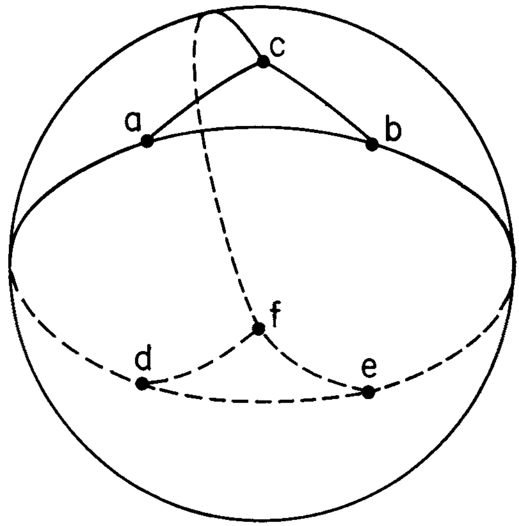

Fig. 1

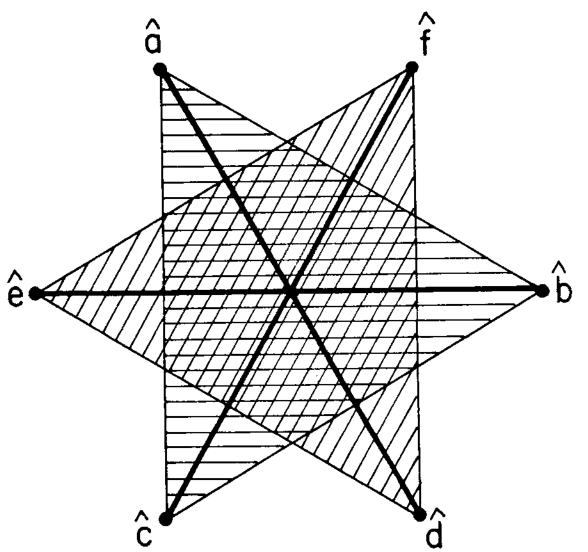

Fig. 2

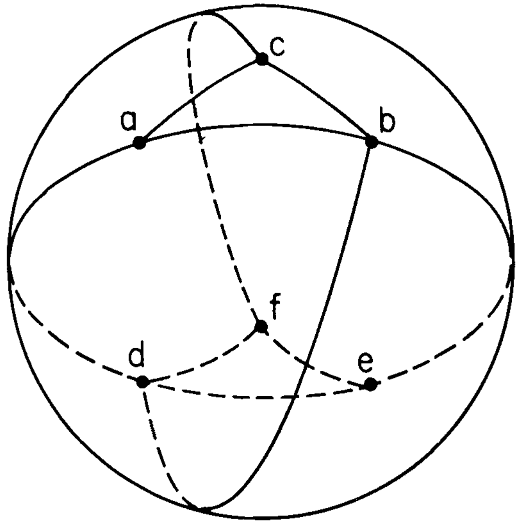

Fig. 3

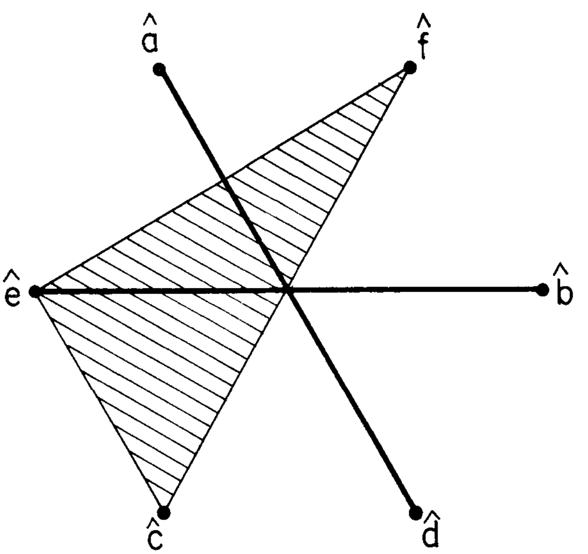

Fig. 4

Theorem (3) can be extended to give a criterion for a spherical complex to be the radial projection of a star-shaped polytope. Details will be published.

References

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